Three-dimensional Analytical Calculation of the Magneto-optical Trapping Forces on a Stationary $J = 0 \rightarrow J = 1$ Atom

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Abstract

The response of a stationary atom with a $J = 0$ ground state and a $J = 1$ excited state is calculated for arbitrary light intensities and polarizations in the presence of a small magnetic field. The force on the atom is calculated in a coordinate-system-independent manner, so the result is valid for three-dimensional superpositions of light fields. Two types of forces are obtained, the first accounting for the usual MOT force using circularly polarized light beams and the second giving the "rectified" MOT force that arises from linearly polarized light beams. New contributions from beams that are neither co- nor counter-propagating are identified. As expected from previous work, for the rectified MOT the force does not saturate for large intensities.

1 Introduction

Since the proposal of spontaneous-force trapping of neutral atoms [1] the magneto-optical trap (MOT) [2] and its variants [3, 4, 5] have been extensively used for three-dimensional optical trapping of atoms. As explained in Ref. [2], the operation of the MOT takes advantage of the magnetic-field-induced circular dichroism of the atoms. For light fields tuned to the low frequency side of the atomic resonance, the circular dichroism favors absorption of photons whose spins are anti-parallel to the local magnetic field (see Fig. 1). Using a magnetic field that changes direction around the equilibrium position of the trap and counterpropagating light fields of opposite spin, a stable trap is produced.

It is straightforward to construct a simple rate equation model of a MOT in one dimension[6]. Such models give reasonable agreement with experiment for low intensities, but cannot be expected to accurately predict saturation or light-polarization properties of the trap. The forces depend strongly on optical and Zeeman coherences that are produced by the light fields and the magnetic field—features that require the full Optical Bloch Equations (OBEs) for proper treatment. This paper presents analytical, fully three-dimensional results for the magnetic-field-dependent forces on atoms that have zero total angular momentum in the ground state and angular momentum $J = 1$ in the excited state. The results are valid for small values of the magnetic field.

The key to the calculation is the observation that, for stationary atoms in sufficiently small magnetic fields, the atomic response is dominated by the local electric field of the light. Thus the calculation is made with an atomic basis that naturally and simply accounts
2 Atomic Response to the Light Field and a Weak Magnetic Field

The system under consideration is a stationary atom with a ground state $|g\rangle$ of zero angular momentum, and an excited state of angular momentum $J = 1$ and energy $\hbar \omega_0$, interacting with the local electric field of the light $E \exp(-i \omega t) + c.c.$ and a static magnetic field $B$. Without sacrificing any generality, the electric field is taken to lie in the $x - y$ plane of a local Cartesian coordinate system, and so can be written in terms of spherical basis vectors[8] as $E = E(-e_+ \hat{r}_- - e_- \hat{r}_+)$, where $E = |E|^2$.

The electric-dipole Hamiltonian in the rotating wave approximation is $V = e r \cdot E e^{-i \omega t}$. If we use for the basis states of the atom the Zeeman levels labelled by the customary
that is necessary to calculate the induced dipole moment and hence the force: 

For zero magnetic field, however, the problem can be simplified to a two-level system if we take as the basis states superpositions of \(|±1\rangle\) constructed such that \(V\) has a non-zero matrix element for only one of them. One choice for the superpositions is \(|e\rangle = e_{-1}|1\rangle + e_{+1}|−1\rangle\) and \(|e'\rangle = −e_{+1}^*|1\rangle + e_{-1}^*|−1\rangle\). With these definitions the electric dipole matrix elements become \(\langle e|er\cdot E|g\rangle \equiv \hbar e\) and \(\langle e'|er\cdot E|g\rangle = 0\), where \(e\) is the Rabi frequency. Therefore the light only couples \(|g\rangle\) and \(|e\rangle\) and the OBE’s are identical to the two-level case. In the absence of a magnetic field the steady-state density matrix elements are found to be

\[
\rho_{ee}(0) = \frac{e^2}{\Delta^2 + \Gamma^2/4 + 2e^2} \tag{1}
\]

\[
\rho_{eg}(0) = \frac{\sigma_e(0) \exp(−i\omega t)}{\Delta^2 + \Gamma^2/4 + 2e^2} \exp(−i\omega t), \tag{2}
\]

where \(Δ = ω − ω_0\) is the detuning of the light field from resonance and \(1/\Gamma\) is the natural lifetime of the excited state. For reference, the Rabi frequency is related to \(Γ\) and the local light intensity \(I\) via \(I/Is = 8e^2/Γ^2 = 3λ^3I/2π^2hcΓ\), where \(Is\) is the saturation intensity.

With the addition of a magnetic field, the Zeeman interaction \(\hat{g}_j\mu_B\hat{B}\cdot\mathbf{J} \equiv \hbar\Omega\cdot\hat{\mathbf{J}}\) couples \(|e\rangle\) to \(|e'\rangle\) and \(|0\rangle\). The density matrix elements then depend on the Larmor frequency \(Ω\). To first order in \(Ω\), the only additional density matrix elements are the Zeeman coherences \(ρ_{0e}\) and \(ρ_{e'e}\) and the optical coherences \(ρ_{0g} \equiv σ_0 \exp(−i\omega t)\) and \(ρ_{e'g} \equiv σ_{e'} \exp(−i\omega t)\). The relevant OBEs are therefore

\[
\dot{\rho}_{ee}(\Omega) = −i\Gammaρ_{ee}(\Omega) + e(σ_e^*(\Omega) − σ_e(\Omega)) \tag{3}
\]

\[
i\dot{σ}_e(\Omega) = −(Δ + iΓ/2)σ_e(Ω) + \epsilon(1 − 2ρ_{ee}(Ω)) + \langle e|\Omega\cdot J|e\rangle σ_e(0) \tag{4}
\]

\[
\dot{\rho}_{0e}(\Omega) = −iΓρ_{0e}(\Omega) + \langle 0|Ω\cdot J|e\rangle ρ_{ee}(0) − \epsilonσ_0(Ω) \tag{5}
\]

\[
i\dot{σ}_0(Ω) = −(Δ + iΓ/2)σ_0(Ω) + \langle 0|Ω\cdot J|e\rangle σ_e(0) − \epsilonρ_{0e}(Ω) \tag{6}
\]

\[
\dot{\rho}_{e'e}(\Omega) = −iΓρ_{e'e}(Ω) + \langle e'|Ω\cdot J|e\rangle ρ_{ee}(0) − \epsilonσ_{e'}(Ω) \tag{7}
\]

\[
\dot{σ}_{e'}(Ω) = −(Δ + iΓ/2)σ_{e'}(Ω) + \langle e'|Ω\cdot J|e\rangle σ_e(0) − \epsilonρ_{e'e}(Ω) \tag{8}
\]

The steady-state solutions to the above are easily solved since they are actually three pairs of coupled equations. Only the results for the \(σ(Ω)\)'s are given below since they are all that is necessary to calculate the induced dipole moment and hence the force:

\[
σ_e(Ω) = σ_e(0) \left(1 + \langle e|Ω\cdot J|e\rangle \left(\frac{2\Delta}{\Delta^2 + Γ^2/4 + 2e^2} − \frac{1}{\Delta − iΓ/2}\right)\right) \tag{9}
\]

\[
σ_0(Ω) = \langle 0|Ω\cdot J|e\rangle σ_e(0) \left(1 + \frac{iε^2/Γ}{\Delta − iΓ/2}\right) \tag{10}
\]

\[
σ_{e'}(Ω) = \langle e'|Ω\cdot J|e\rangle σ_e(0) \left(1 + \frac{iε^2/Γ}{\Delta − iΓ/2}\right) \tag{11}
\]

The matrix elements of \(Ω\cdot J\) are easily calculated, and are

\[
\langle e|Ω\cdot J|e\rangle = Ω_z(|e_{-1}|^2 − |e_{+1}|^2) = iΩ_z(\hat{E}×\hat{E}^*) \tag{12}
\]

\[
\langle 0|Ω\cdot J|e\rangle = Ω_{+1}e_{-1} + Ω_{−1}e_{+1} \tag{13}
\]

\[
\langle e'|Ω\cdot J|e\rangle = −2Ω_{e_{+1}}e_{−1}. \tag{14}
\]
Note that in Eq. 12 the matrix element is expressed as a triple product, i.e. without reference to any particular coordinate system. This practice emphasizes the three-dimensional nature of the results, and will be continued in the following.

The induced dipole moment \( \mathbf{p} = -eTr(\mathbf{rp}) \) can now be determined. The calculation is simplified by the use of the identity

\[
\mathbf{r} = z\mathbf{\hat{z}} + \mathbf{r} \cdot \mathbf{\hat{E}}^* - \mathbf{z} \cdot (\mathbf{r} \times \mathbf{\hat{E}}) \mathbf{\hat{z}} \times \mathbf{\hat{E}}^*
\]

in which case

\[
\mathbf{p} = -e \left( \langle g|\mathbf{r} \cdot \mathbf{\hat{E}}^*|e \rangle \sigma_e(\Omega)\mathbf{\hat{E}} + \langle g|z|0\rangle \sigma_0(\Omega)\mathbf{\hat{z}} - \langle g|\mathbf{z} \cdot (\mathbf{r} \times \mathbf{\hat{E}})|e'\rangle \mathbf{\hat{z}} \times \mathbf{\hat{E}}^* \right)
\]

The result for the magnetic-field-dependent part of the dipole moment is

\[
\mathbf{p} = \frac{\lambda}{8\pi^2} \left[ \sigma_1(\Delta, \epsilon^2)\Omega \times \mathbf{E} + \sigma_2(\Delta, \epsilon^2)\Omega \cdot (\mathbf{\hat{E}}^* \mathbf{E}) \right]
\]

The functions \( \sigma_{1,2}(\Delta, \epsilon^2) \) are given by

\[
\sigma_1(\Delta, \epsilon^2) = \frac{3\lambda^2\Gamma}{4\pi} \left( \frac{-\Delta \Gamma - i(\Delta^2 - \Gamma^2/4 + \epsilon^4/\Gamma^2)}{(\Delta^2 + \Gamma^2/4 + 2\epsilon^2)(\Delta^2 + (\Gamma/2 + \epsilon^2/\Gamma)^2)} \right)
\]

\[
\sigma_2(\Delta, \epsilon^2) = \frac{3\lambda^2\epsilon^2}{4\pi} \left( \frac{\Delta(\Gamma^2 - \epsilon^2) + i(3\Delta^2\Gamma + \Gamma^3/4 + 5\Gamma\epsilon^2/2 + 4\epsilon^4/\Gamma)}{(\Delta^2 + \Gamma^2/4 + 2\epsilon^2)^2(\Delta^2 + (\Gamma/2 + \epsilon^2/\Gamma)^2)} \right)
\]

and are interpreted as photon scattering cross-sections per unit Larmor frequency. The \( \Omega \times \mathbf{E} \) part of \( \mathbf{p} \) can be easily understood as resulting from the Lorentz force \( \dot{\mathbf{p}} \times \mathbf{B}/c \propto \mathbf{E} \times \mathbf{B} \).

### 3 Magneto-Optical Forces

The light-force on the atom is given by \( \mathbf{F} = \mathbf{p} \cdot \nabla \mathbf{E}^* + \text{c.c.} \). This in general has terms that depend on intensity gradients as well as scattering forces. Since most experiments use collimated Gaussian light beams for the trapping, the intensity gradients can be neglected. The total electric field can then be assumed to be produced by the superposition of a number of nearly plane waves, travelling in various directions \( \mathbf{k} \):

\[
\mathbf{E} = \sum_k \mathbf{E}_k \exp(i\mathbf{k} \cdot \mathbf{r}).
\]

Then the force becomes

\[
\mathbf{F} = -i \sum_k \mathbf{k} \mathbf{p} \cdot \mathbf{E}_k \exp(-i\mathbf{k} \cdot \mathbf{r}) + \text{c.c.}
\]

Note that the force produced by one beam is in general affected by the dipole produced by the other beams.

The force that arises from the first term in Eq. 17 dominates at low intensities. It is given by

\[
\mathbf{F}_1 = -i \sum_k \frac{\mathbf{k}}{4\pi} \sigma_1(\Omega) \cdot (\mathbf{E} \times \mathbf{E}_k^*) \exp(-i\mathbf{k} \cdot \mathbf{r}) + \text{c.c.}
\]
This simplifies to

\[ F_{\text{MOT}} = - \sum_k \hat{\mathbf{k}} \frac{\sigma_1 I_k}{2c} \mathbf{\Omega} \cdot \mathbf{s}_k + \text{c.c.} \tag{23} \]

if \( \sigma_1 \) is assumed to have no spatial variation, and rapidly spatially varying terms are dropped. The photon spin for beam \( \mathbf{k} \) is given by \( \mathbf{s}_k = i \mathbf{E}_k \times \mathbf{E}_k^* \). For negative detunings, \( \sigma_1 \) is positive, so \( F_{\text{MOT}} \) is in the direction of \( \hat{\mathbf{k}} \) when the photon spin is antiparallel to the magnetic field. This is as expected from the simple considerations of Ref. [2] and Fig. 1. Note that \( F_{\text{MOT}} \) vanishes for linearly polarized light.

It is important to appreciate that ignoring the spatial dependence of \( \sigma_1 \), such as was done in obtaining Eq. 23, can lead to significant errors at high intensities. This is because the intensity dependences of \( \sigma_1 \) and \( \sigma_2 \) (Eq.’s 18 and 19) spatially modulate the force if an intensity gradient exists as must always be the case for three-dimensional traps. This can produce a “rectification” [9] of a force that would average to zero if only the explicit electric field dependences from Eq. 22 were considered.

The second term in Eq. 17 depends on the local photon spin \( \mathbf{s} = i \mathbf{E} \times \mathbf{E}^* \):

\[ F_2 = - \sum_k \frac{\hat{k}}{4\pi} \sigma_2 \mathbf{\Omega} \cdot \mathbf{s} \mathbf{E} \cdot \mathbf{E}_k^* \exp(-i\mathbf{k} \cdot \mathbf{r}) + \text{c.c.} \tag{24} \]

This force depends on the component of the magnetic field along the direction of the local photon spin. For small intensities, it is proportional to \( I^2 \) and hence is negligible compared to \( F_1 \), but for large intensities it becomes dominant.

As an application of these formulas (22 and 24), consider a 1-D MOT produced by two counterpropagating beams of equal intensity but opposite spin (\( \mathbf{s}_k = \pm \hat{\mathbf{z}} \)). In this case the light polarization is linear but changes direction at each point in space. Thus the local spin \( i \mathbf{E} \times \mathbf{E}^* \) vanishes, giving \( F_2 = 0 \). In addition, for this simple case, there are no intensity gradients so rectification of the rapidly oscillating terms of Eq. 22 does not occur and the force is given entirely by Eq. 23, which simplifies to

\[ F_{\text{MOT}} = - \frac{\text{Re}(\sigma_1) I \Omega_z}{c} \hat{\mathbf{z}}, \tag{25} \]

where the propagation axis is taken to be the z-axis, and the total intensity of the two beams is \( I \). For large intensities, the force decreases as \( I^{-2} \) in contrast to \( I^{-1} \) as expected from the simple model mentioned above [6]. For a 3-D MOT, this formula holds for small intensities, but for large intensities rectification must be taken into account.

A second situation of interest is a pair of counterpropagating linearly polarized beams. In this case there are intensity gradients and so rectification must be taken into account for high intensities. Both \( F_1 \) and \( F_2 \) contribute to the force, with the result

\[ \mathbf{F} = \frac{\Omega_z I \sin \theta}{c} \left( \text{Im}(\sigma_1) \cos 2kz - \frac{\text{Im}(\sigma_2) \cos \theta \sin^2 2kz}{1 + \cos \theta \cos 2kz} \right) \hat{\mathbf{z}} \tag{26} \]

where \( \theta \) is the angle between the polarizations of the two beams. For small intensities only the \( \sigma_2 \) term contributes, with the spatially averaged final result being

\[ \mathbf{F} = -\hbar k \Omega \sin \theta \cos \theta \left( \frac{I}{I_s} \right)^2 \frac{1 + 12\Delta^2/\Gamma^2}{(1 + 4\Delta^2/\Gamma^2)^3} \hat{\mathbf{z}} \tag{27} \]
The $\sin \theta \cos \theta$ is consistent with the dependence observed in Ref. [5]. Note that the force is proportional to $I^2$, indicating the stimulated nature of the force.

For high intensities, the spatially dependent force is

$$F = -\hbar k \Omega \sin \theta \left( \frac{2 \cos \theta \sin^2 2kz}{(1 + \cos \theta \cos 2kz)^2} + \frac{\cos 2kz}{1 + \cos \theta \cos 2kz} \right) \hat{z} \tag{28}$$

Note that this is independent of intensity. The spatial average is $F = -\hbar k \Omega \tan \theta (1 - 1/|\sin \theta|)$, except for small angles. This is in agreement with the numerical calculations of Ref. [5] for $\theta = \pi/4$. This formula is valid as long as $\Omega \ll \epsilon$, and shows that if $\epsilon \gg \Gamma$ then the scattering force limit $\hbar k \Gamma/2$ can be exceeded, as has been pointed out elsewhere [5, 9].

4 Conclusions

Explicit analytical expressions for the magnetic-field-induced light forces on a $J = 0 \rightarrow J = 1$ atom were obtained for arbitrary intensities and polarizations of the light waves in the limit of small magnetic field. The results allow for 3-D configurations of light waves and explicitly show for the first time the existence of magnetic-field-dependent forces generated by inter-beam couplings.

The methods used here can be extended to more complex atomic configurations. In particular, the steady-state atomic response in the absence of the magnetic field can be readily obtained. For example, for a $1/2 \rightarrow 3/2$ configuration, there are two superpositions of states analogous to $|e\rangle$, one for each of the two ground state sublevels. Simple equations can then be written that allow for repopulation pumping. The situation becomes slightly more complex for $J = 1 \rightarrow J = 2$, where a linear superposition of the $\pm 1$ ground-state sublevels is found by diagonalizing a $2 \times 2$ matrix. The advantage of this method is that the optically produced Zeeman coherences are built into the basis states, and the optical coherences are simply related to the ground-state populations. Then the ground-state populations are easily determined by balancing incoming and outgoing spontaneous emission rates. This procedure is completely equivalent to solving the full OBE’s in the absence of a magnetic field.

Finally, it should be noted that a similar approach can be used to find the dark-state resonances for $J \rightarrow J$ and $J \rightarrow J - 1$ configurations. It is possible to construct an explicit formula to determine the necessary superpositions of Zeeman levels for which there are no couplings to any excited states.

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References
